Fakultet za proizvodnju i menadžment Trebinje

Matematika III

Drugi kolokvijum

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Ime i prezime, broj indeksa:______________ Poeni:						

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1. a) Uvodeći cilindrične koordinate

$$I = \int\int\int_V z\sqrt{x^2+y^2}dxdydz,$$

ukoliko je oblast $V\,: x^2+y^2 \leq 2x,\, y \geq 0, 0 \leq z \leq a.$

Rješenje:

$V\,: x^2+y^2 \leq 2x,\, y \geq 0, 0 \leq z \leq a$

$(x-1)^2+y^2=1$

$$I = \int\int\int_V z\sqrt{x^2+y^2}dxdydz = \begin{bmatrix} \text{Cilindrične koordinate} \\ x=\rho\cos\varphi, y=\rho\sin\varphi, z=z, |J|=\rho \\ \rho \leq 2\cos\varphi, 0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq z \leq a \end{bmatrix} =$$

$$= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} d\rho \int_0^a z\rho^2 dz = \frac{a^2}{2}\int_0^{\frac{\pi}{2}} \frac{8}{3}\cos^3\varphi d\varphi =$$

$$= \frac{4a^2}{3}\int_0^{\frac{\pi}{2}} \cos\varphi(1-\sin^2\varphi)d\varphi = \frac{4a^2}{3}(u-\frac{u^3}{3})\mid_0^{\frac{\pi}{2}} = \cdots = \frac{8a^2}{9}.$$

b) Izračunati zapreminu elipsoida

$$\frac{x^2}{16}+\frac{y^2}{25}+\frac{z^2}{9}=1.$$

Rješenje:

$$V = \int\int\int_T dxdydz,\ \text{pri čemu je } T \text{ elipsoid.}$$

Uvodimio sferne koordinate:

$$x = 4\rho\cos\varphi\sin\theta$$
$$y = 5\rho\sin\varphi\sin\theta$$
$$z = 3\rho\cos\theta$$
$$J = 60\rho^2\sin\theta$$
$$T' = \{(\rho,\varphi,\theta) : 0 \leq \rho \leq 1, 0 \leq \varphi \leq 2\pi, 0 \leq \theta \leq \pi\}.$$

$$V = \int\int\int_{T'} 60\rho^2\sin\theta d\rho d\varphi d\theta = 60\int_0^1 \rho^2 d\rho \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta = \cdots = 80\pi.$$

2. Izračunati keivolinijski integral prve vrste $\oint_{(K)} ye^{-x}ds$ gdje je kriva (K) zadana u parametarskom obliku jednačinama $x(t) = \ln(1+t^2), y(t) = 2\arctan t - t + 3$ između tačaka $t = 0$ i $t = 1$.

Rješenje:

$$\oint_{(K)} f(x,y)ds = \int_a^b f(\varphi(t), \psi(t))\sqrt{\varphi'^2(t) + \psi'(t)}dt,$$

$$x'^2 + y'^2 = \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2 = \cdots = 1,$$

$$f(\varphi(t), \psi(t)) = (2\arctan t - t + 3)e^{-\ln(1+t^2)} = \frac{2\arctan t - t + 1}{1+t^2}.$$

$$\oint_K ye^{-x}ds = \int_0^1 \frac{2\arctan t - t + 1}{1+t^2}dt = \int_0^1 \frac{2\arctan t}{1+t^2}dt - \int_0^1 \frac{t}{1+t^2}dt + \int_0^1 \frac{3}{1+t^2}dt =$$

$$= I_1 - I_2 + I_3.$$

$$I_1 = \frac{2\arctan t - t + 1}{1+t^2} = \left[\arctan t = u, \frac{1}{1+t^2}dt = du\right] = \int_0^1 2udu = \arctan^2 t \mid_0^1 = \left(\frac{\pi}{2}\right)^2.$$

$$I_2 = \int_0^1 \frac{t}{1+t^2}dt = \left[1+t^2 = u, 2tdt = du\right] = \int_0^1 \frac{1}{u}\frac{du}{2} = \frac{1}{2}ln(1+t^2) \mid_0^1 = \frac{1}{2}\ln 2.$$

$$\int_0^1 \frac{3}{1+t^2}dt = 3\arctan t \mid_0^1 = 3 \cdot \frac{\pi}{4}.$$

3. a) Naći krivolinijski integral druge vrste $\int_l xydx + (x+y)dy$ gdje je (l): (a) Ossječak prave $y = x$ od $O(0,0)$ do tačke $A(1,1)$, (b) luk parabole $y = x^2$ od O do A.

Rješenje:

(a) $y = x, dy = dx$, $\int_l xydx + (x+y)dy = \int_0^1 (x^2 + 2x)dx = \frac{4}{3}$.

(b) $y = x^2, dz = 2xdx$, $\int_l xydx + (x+y)dy = \int_0^1 (3x^3 + 2x^2)dx = \frac{17}{12}$.

b) Pomoću Grinove formule izračunati integral $\int_C (xy + x + y)dx + (xy + x - y)dy$ ako je C kontura pozitvno orijentisane kružnice $x^2 + y^2 = ax$.

Rješenje:

$P(x,y) = xy + x + y; \frac{\partial P}{\partial y} = x + 1,$

$Q(x,y) = xy + x - y, \frac{\partial Q}{\partial x} = y + 1.$

$C : (x - \frac{a}{2})^2 + y^2 = \frac{a^2}{4}.$

$$I = \int\int_S (y-x)dxdy = \left[\begin{array}{c} \text{Polarne koordinate} \\ x = \frac{a}{2} + \rho\sin\varphi; y = \rho\cos\varphi, J = \rho, \\ 0 \le \rho \le \frac{a}{2}, 0 \le \varphi \le 2\pi \end{array}\right] =$$

$$= \int_0^{\frac{a}{2}} d\rho \int_0^{2\pi} (\rho\cos\varphi - \rho\sin\varphi - \frac{a}{2})\rho d\varphi = \int_0^{2\pi} d\varphi \left(\frac{\rho^3}{3}(\cos\varphi - \sin\varphi) - \frac{a}{2}\frac{\rho^2}{2}\right) \mid_0^{\frac{a}{2}} =$$

$$= \int_0^{2\pi} \left[\frac{a^3}{24}(\cos\varphi - \sin\varphi) - \frac{a^3}{16}\right] d\varphi = \cdots = \frac{-a^3\pi}{8}.$$

4. Izračunati površinski integral prve vrste $I = \int\int_S (x^2+y^2)dS$ gdje je S dio konusne površi $z^2 = x^2+y^2$, $0 \le z \le 1$.

Rješenje:

$z^2 = x^2 + y^2, z = \sqrt{x^2+y^2}$

$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$,

$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$.

$I = \int\int_D (x^2+y^2)\sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}dxdy$

$D = \{(x,y) : x^2+y^2 \le 1\}$

Uvedimo polarne koordinate $x = \rho\cos\varphi, y = \rho\sin\varphi$ tako da $D \to D'$, pri čemu je $D' = \{(\rho,\varphi) : 0 \le \rho \le 1, 0 \le \varphi \le 2\pi\}$.

$$I = \int_0^{2\pi} d\varphi \int_0^1 \rho^3\sqrt{2} = \cdots = \frac{\pi\sqrt{2}}{2}.$$

5. Primjenom teorema Gausa-Ostrogradskog izračunati

$$\int\int_S 3dxdy + 2ydxdz - x^2zdydz,$$

ako je S površ koja ograničava tijelo $V : 4x^2 + y^2 + 4z^2 \le 1, x \ge 0, y \ge 0, z \ge 0$ orijentisana vektorom vanjske normale.

Rješenje:

$\frac{\partial P}{\partial x} = -2xz$,

$\frac{\partial Q}{\partial y} = 2$,

$\frac{\partial R}{\partial z} = 0$.

Sferene koordinate

$$\begin{array}{ll} x = \frac{1}{\rho}\cos\varphi\sin\theta & 4x^2+y^2+4z^2 \le 1, x \ge 0, y \ge 0, z \ge 0 \\ y = \rho\sin\varphi\sin\theta & \vdots \\ z = \frac{1}{2}\rho\cos\theta & \rho \le 1 \\ J = \frac{1}{4}\rho^2\sin\theta & V' = \{(\rho,\varphi,\theta) : 0 \le \rho \le 1, 0 \le \varphi \le \frac{\pi}{2}, 0 \le \theta \le \frac{\pi}{2} \end{array}$$

$$I = \int\int\int_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz = \frac{1}{4}\int_0^1 d\rho \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \left(\frac{-1}{2}\rho^2\cos\varphi\sin\theta\cos\theta + 2\right)\cdot\rho^2\sin\theta d\theta =$$

$$= \cdots = \frac{-1}{120} + \frac{5\pi}{24}.$$